A Semi-analytical Model for Estimation of the Transfer Function of a Coupled Resonator Optical Waveguide Based on Coupled Mode Theory

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Abstract: *Coupled Resonator Optical Waveguides (CROWs) are receiving increased attention for the realization of future nanophotonic waveguiding devices. In this paper a detailed analysis is provided in order to study the frequency response of finite CROWs including the input/output waveguide-cavity coupling. A semi-analytical model is obtained based on the application of reciprocity relations on Maxwell's equations and can also include the coupling between non adjacent cavities and handle CROWs of arbitrary long length. The model is verified by comparing its results to the Finite Difference Time Domain (FDTD) method.*

Introduction

Coupled resonator optical waveguides (CROWs) present a new type of light-waveguiding by propagation through coupling between adjacent unit cells of a nanophotonic optical integrated circuit [1]. Such devices are currently a topic of intense research due to their potential applications in optical filtering, dispersion compensation, optical buffering, nonlinear optics, etc [2]-[6]. One possible realization of a CROW is a sequence of coupled defect cavities embedded in a two-dimensional periodic photonic crystal structure. These defect cavities are designed such that their resonant frequency falls within the forbidden gap of the surrounding 2D photonic crystal structure and thus permitting coupling due to the evanescent Bloch waves [4].
The frequency response of the CROW also depends on how light is coupled into the structure. Figure 1 illustrates one possible arrangement where light propagating in a waveguide is side coupled to the first cavity of the CROW and another waveguide collects the light from the last cavity. Such structures can be analysed either with the finite difference time domain method (FDTD) using direct numerical solution of Maxwell's equations. However, as the number of cavities becomes larger, FDTD simulations eventually become impractical due to the large computational time required. In addition FDTD simulations usually do not provide any useful physical insight. To properly engineer a CROW, one rather needs a fast analytical or semi-analytical model. Such models already exist based on the tight binding approximation, for both finite and infinite CROWs but without taking into account the coupling between the cavities and the input/output waveguides. One can use coupled mode theory (CMT) to extend the model in order to include the influence of the waveguides.

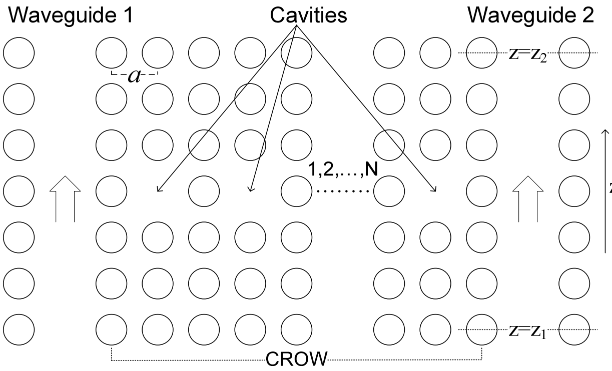

Figure 1: CROW device containing input waveguide 1 and output waveguide 2 and N cavities.

In this paper a detailed derivation of the coupled mode equations relating the amplitudes of the cavity and waveguide modes is given for the CROW/waveguide system starting from Maxwell's equations and using the reciprocity relations. This provides a useful, semi analytical model for the efficient estimation of the transfer function of a device with an arbitrary number of cavities. The results of the model are compared with the results of the FDTD method and good agreement is observed.

Field Expansion

In the spirit of the tight binding approximation, in the frequency domain one may expand the total electric and magnetic field **E** and **H** inside the CROW in terms of the modes of the input/output waveguide and the isolated cavities, i.e.

$$\mathbf{E} = \sum_n a_n \mathbf{E}_n + \sum_l a_{fl} \mathbf{E}_{fl} + \sum_l a_{bl} \mathbf{E}_{bl} \qquad (1)$$

$$\mathbf{H} = \sum_n a_n \mathbf{H}_n + \sum_l a_{fl} \mathbf{H}_{fl} + \sum_l a_{bl} \mathbf{H}_{bl} \qquad (2)$$

where $\mathbf{E}_n$, $\mathbf{H}_n$ are the electric and magnetic modal fields of the n^{th} isolated cavity modes ($1 \le n \le N$), ($\mathbf{E}_{fl}$, $\mathbf{H}_{fl}$) and ($\mathbf{E}_{bl}$, $\mathbf{H}_{bl}$) are the forward and backward propagating modes of the l^{th} waveguide. In addition a_n, a_{fl} and a_{bl} denote the excitation coefficients of the n^{th} cavity mode and the forward and backward isolated propagating mode of the l^{th} waveguide respectively. Assuming that z is the propagation direction of the forward waveguide modes, a_{fl} and a_{bl} are considered z-dependent, while the cavity mode amplitudes

a_n are assumed not to depend on z [7]. One may use Bloch's theorem in order to express the waveguide mode fields as:

$$\mathbf{E}_{ml}(\mathbf{r}) = \mathbf{e}_{ml}(\mathbf{r})e^{j\beta_{ml}z} \quad (3)$$

$$\mathbf{H}_{ml}(\mathbf{r}) = \mathbf{h}_{ml}(\mathbf{r})e^{j\beta_{ml}z} \quad (4)$$

where assuming an exp(-$j\omega t$) dependence, the propagation constant β_{ml} is positive for the forward (m=f) and negative for the backward (m=b) propagating mode while $\mathbf{e}_{ml}$ and $\mathbf{h}_{ml}$ are periodic vector functions with the same periodicity a as the input/output waveguides. In addition forward and backward modes fulfill certain symmetry relations as shown in [1]. The isolated waveguide modes obey Maxwell's equation in the frequency domain

$$\nabla \times \mathbf{E}_{ml} = j\omega\mu\,\mathbf{H}_{ml} \quad (5)$$

$$\nabla \times \mathbf{H}_{ml} = -j\omega\varepsilon_l(\mathbf{r})\,\mathbf{E}_{ml} \quad (6)$$

where $\varepsilon_l(\mathbf{r})$ is the dielectric constant of the l^{th} waveguide alone. The waveguide modes are assumed normalized so that

$$\eta_m = \int_S (\mathbf{E}^*_{ml} \times \mathbf{H}_{ml} + \mathbf{E}_{ml} \times \mathbf{H}^*_{ml})\,\mathbf{z}\,dS = \pm 1 \quad (7)$$

where η_m=1 for the forward propagating mode (m=f) and η_m=-1 for the backward propagating mode (m=b). In addition the forward and backward propagating modes of each waveguide obey the following orthogonality relations,

$$\int_S (\mathbf{E}^*_{fl} \times \mathbf{H}_{bl} + \mathbf{E}_{bl} \times \mathbf{H}^*_{fl})\,\mathbf{z}\,dS = 0 \quad (8)$$

The cavity modes obey a similar set of equations,

$$\nabla \times \mathbf{E}_n = j\omega_o\mu\,\mathbf{H}_n \quad (9)$$

$$\nabla \times \mathbf{H}_n = -j\omega_o\varepsilon_{cn}\,\mathbf{E}_n \quad (10)$$

where $\varepsilon_{cn}(\mathbf{r})$ is the dielectric constant of n^{th} cavity alone and ω_0 is the mode resonant frequency. For the cavity modes, the electric fields can be chosen purely real $\mathbf{E}_{rn}^* = \mathbf{E}_{rn}$ resulting in a purely imaginary magnetic field, $\mathbf{H}_{rn}^* = -\mathbf{H}_{rn}$.

Waveguide amplitude equations

In this section using the reciprocity relations outlined in [1] and the field equations presented in the previous section, we derive the amplitude equations for the forward and backward propagation modes of the waveguides. Although one could assume that the waveguides interact only with their adjacent cavities such an approximation will not be used in the derivation in order to obtain a model as general as possible. To apply the reciprocity relations, we first define the vector functions

$$\mathbf{F}_{ml} = \mathbf{E} \times \mathbf{H}^*_{ml} + \mathbf{E}^*_{ml} \times \mathbf{H} \quad (11)$$

Using the 2D form of Green's theorem as in [1], one obtains

$$\frac{\partial}{\partial z}\int_S \mathbf{F}_{ml}\mathbf{z}\,dS = j\omega\int_S (\varepsilon - \varepsilon_l)\mathbf{E}^*_{ml}\mathbf{E}\,dS \quad (12)$$

where the surface S is any plane perpendicular to the z axis. Substituting (1) and (2) in (12) and neglecting the self coupling of the waveguide modes as in [1], leads to the following system of equations for the forward waveguide amplitudes

$$\frac{\partial a_{fl}}{\partial z} = j\mathbf{k}_{fl} \cdot \mathbf{a} \quad (13)$$

where $\mathbf{a}=(a_1,\ldots,a_N)$ and the vector $\mathbf{k}_{fl}$ contains the coupling coefficients of the cavity and waveguide modes, given by

$$k_{fl,n} = \int_S (\omega\varepsilon - \omega_o\varepsilon_{cn})\mathbf{E}^*_{fl}\mathbf{E}_{rn}dS + \mu(\omega - \omega_o)\int_S \mathbf{H}^*_{fl}\mathbf{H}_{rn}dS \quad (14)$$

Using similar arguments one may also derive a system of equations for the backward amplitudes,

$$\frac{\partial a_{bl}}{\partial z} = -j\mathbf{k}_{bl} \cdot \mathbf{a} \quad (15)$$

where

$$k_{bl,n} = \int_S (\omega\varepsilon - \omega_o\varepsilon_{cn})\mathbf{E}^*_{bl}\mathbf{E}_n dS + \mu(\omega - \omega_o)\int_S \mathbf{H}^*_{bl}\mathbf{H}_n dS \quad (16)$$

These equations relate the rate of change of the waveguide mode excitation coefficients to the cavity amplitudes and will be used to derive simpler relations between the cavity amplitudes and the waveguide excitation coefficients.

Cavity amplitude equations

The derivation of the cavity amplitude equation is similar to the one used above for the waveguides. The vector functions defined are:

$$\mathbf{F}_n = \mathbf{E} \times \mathbf{H}^*_n + \mathbf{E}^*_n \times \mathbf{H} \quad (17)$$

and from applying the 3D version of Green's theorem and taking into account that the cavity modal fields vanish at infinity, we obtain

$$j\mu(\omega - \omega_o)\int_V \mathbf{H}^*_n\mathbf{H}\,dV + j(\omega\varepsilon - \omega_o\varepsilon_{cn})\int_V \mathbf{E}^*_n\mathbf{E}dV = 0 \quad (18)$$

After substituting the total field expressions from (1) and (2) we obtain the coupled mode equation for the amplitudes of the cavities

$$j\int_{z_1}^{z_2}\mathbf{k}_{fl}^{*}\cdot\mathbf{a}\,dz+j\int_{z_1}^{z_2}\mathbf{k}_{bl,n}^{*}\cdot\mathbf{a}\,dz+j\boldsymbol{K}\mathbf{a}=0 \quad (19)$$

where the elements of the cavity mode coupling matrix are given by

$$K_{pq}=\int_V \mu(\omega-\omega_o)\mathbf{H}_p^{*}\mathbf{H}_q dV+ \int_V (\omega\varepsilon-\omega_o\varepsilon_p)\mathbf{E}_p^{*}\mathbf{E}_q dV \quad (20)$$

Assuming that the cavity modes are normalized to have unity energy, the diagonal elements of $\boldsymbol{K}$ which represent the self coupling coefficients can be simplified by neglecting the cavity mode self coupling in the waveguides and thus obtaining $K_{mm}=\omega-\omega_0$. Applying integration by parts, transforms equation (19) to

$$\sum_l a_{fl}(z_1)\mathbf{K}_{fl}+\sum_l a_{bl}(z_2)\mathbf{K}_{bl}+\boldsymbol{J}\mathbf{a}+j\boldsymbol{K}\mathbf{a}=0 \quad (21)$$

where

$$J_{nl}=\int_{z_1}^{z_2}(k_{fl,n}\Lambda_{fl,n}^{*}-k_{bl,n}\Lambda_{bl,n}^{*})dz \quad (22)$$

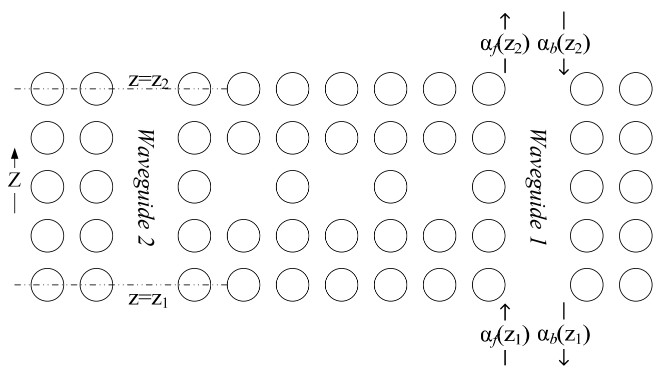

Figure 2: Simplified 3-cavity and 2-waveguide CROW device.

where $\Lambda_{ml,n}$ and are given by

$$\Lambda_{ml,n}=\int_{z_1}^{z_2}k_{ml,n}dz \quad (23)$$

and $\mathbf{K}_{fl}$ and $\mathbf{K}_{bl}$ are equal to $-j\Lambda_{fi}^{*}(-l)$ and $j\Lambda_{bi}^{*}(l)$ respectively.

Transfer Function Calculation

In order to obtain the transmission transfer function $H(\omega)$ of the CROW, one can use equation (21) combined with equation (13). Alternatively the reflection transfer function can be obtained using (21) and (15). Assuming waveguide 1 is the input waveguide then

$$H(\omega)=\frac{a_{f1}(z_2)}{a_{f1}(z_1)}e^{j\beta L} \quad (24)$$

where $L=z_2-z_1$. Assuming no backward propagating modes at both waveguides at $z=z_2$ and no forward propagating mode at waveguide 2 at $z=z_1$ then (21) yields

$$\mathbf{a}=-\left[(\boldsymbol{J}+j\boldsymbol{K})^{-1}\mathbf{K}_{f1}\right]a_{f1}(z_1) \quad (25)$$

Using (13), one obtains

$$\frac{\partial a_{f1}}{\partial z}=-ja_{f1}(z_1)\int_{z_1}^{z_2}\mathbf{k}_{f1}(z)\left[(\boldsymbol{J}+j\boldsymbol{K})^{-1}\mathbf{K}_{f1}\right]dz \quad (26)$$

Integrating with respect to z and using (24),

$$H(\omega)=\left(1-\left[(\boldsymbol{J}+j\boldsymbol{K})^{-1}\mathbf{K}_{f1}\right]\mathbf{K}_{f1}\right)e^{j\beta L} \quad (27)$$

Equation (27), relates the transfer function of the CROW to the coupling coefficients of the cavities and the input/output waveguides which can be computed using numerical integration. In the following subsection the semi-analytical model for the estimation of the transfer function based on (27), will be compared to FDTD simulations in the case of a finite CROW.

Simulation Results

To test the validity of the model, the finite CROW presented in Figure 2 was considered, consisting of three 2D photonic crystal defect cavities and two photonic crystal waveguides. The center to center distance between the lattice rods is r_a=0.6μm while their radius is $r=0.18r_a$. The refractive index of rods equals n_{rods}=3.4 and that of the surrounding medium is n_{air}=1.0. The cavity mode resonant frequency calculated by the plane wave expantion (PWE) was near f_o=195.75THz that corresponds to a resonant wavelength of 1.532μm.

Both the cavity and waveguide mode field distributions were calculated using the PWE method. The cavity mode is normalized so as to have unit total electromagnetic energy while the waveguide mode is normalized according to (7).

Using (27) one can obtain the transfer function for the transmission of waveguide 1. Figure 3 depicts the power transfer function $T=|H|^2$ of the transmission of waveguide 1. The three peaks of T are due to the resonant frequencies of the three cavity system. Furthermore the amplitude of the power transfer function reaches approximately the value 0.25 at the resonant frequencies. In order to test the validity of our results an FDTD simulation with the same parameters was performed and the resulting power transfer function of the transmission of waveguide 1 was calculated. This result is depicted in Figure 4. As expected there still exist three peeks for the tranfer function due to the cavity modes, however one notices a small difference of the order of 1%-2% in the resonant frequencies computed by the two methods. This slight disagreement maybe due to the inherent inaccuracies of the plane wave expansion method used to calculate the modal fields as well as the coupling coefficients

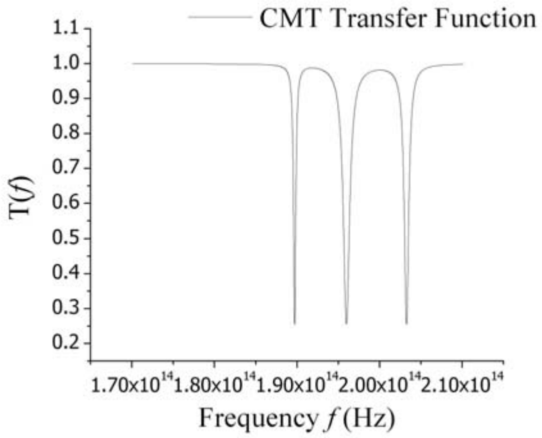

Figure 3: Power tranfer function of the 3-cavity CROW calculated using equation(25).

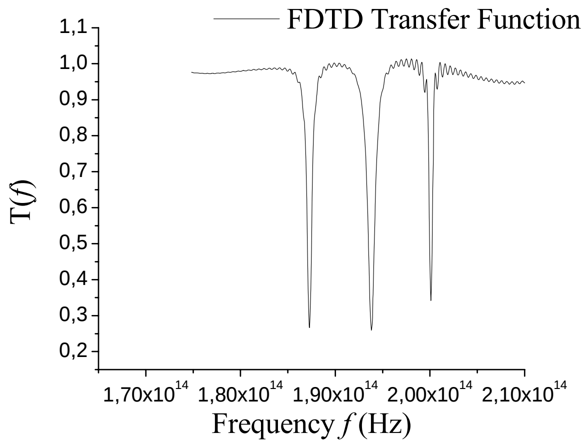

Figure 4: Power tranfer function using FDTD simulations for the 3-cavity CROW.

in (27) and the FDTD methods.

Finally both the FDTD and CMT power transfer functions are plotted in Figure 5 shifted so that the first resonant frequency is the same. One notices good agreement of the amplitude of the first two peeks as well as for their 3dB half-width. A small disagreement is observed for the third peek as a result of possible inaccurancies of the FDTD simulations or the PWE calculations.

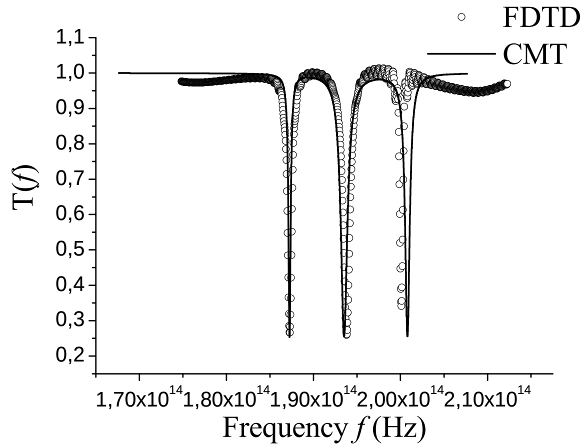

Figure 5: Comparison of FDTD and CMT amplitude power tranfer functions.

Conclusions

This paper presented an analysis for the derivation of a semianalytical model for the estimation of the frequency response of finite CROW devices. Using the reciprocity relations on Maxwell's equations the cavity/waveguide and cavity/cavity coupling coefficients were theoretically calculated. The model was verified using FDTD simulations. The results showed good agreement, however a small difference in the resonant frequencies was detected.

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